GAUSSIAN PROCESSES
EXERCISE SHEET 6: ENTROPY AND CLT(II)

Exercise 1.

As $\hat{X}_n \to N(0,2)$, we have

$$\lim_{n\to\infty} \varphi_{\hat{X}_n}(t) = \lim_{n\to\infty} \varphi_{X_n}(t) e^{-\frac{t^2}{2}} = e^{-t^2}$$

for all $t \in \mathbb{R}$. But this of course implies that $\lim_{n\to\infty} \varphi_{X_n}(t) = e^{-t^2/2}$, so $X_n \to N(0,1)$ in law. $\square$

Exercise 2.

Let $X_1, X_2, \dots$ be i.i.d. random variables with

$$\mathbb{E}[X_i] = 0, \quad \mathrm{Var}(X_i) = 1,$$

and define the normalized sums

$$S_n := \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i.$$

Step 1: Tightness. The variance of S_n is

$$\mathrm{Var}(S_n) = 1,$$

which is uniformly bounded. By Chebyshev's inequality, for any $M > 0$:

$$\mathbb{P}(|S_n| > M) \leq \frac{1}{M^2} \to 0 \quad \text{as } M \to \infty.$$

Hence, the sequence (S_n) is tight, so there always exist subsequences that converge in distribution.

Step 2: Verify Stein's condition. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable with f and f' bounded. Then

$$\mathbb{E}[S_n f(S_n)] = \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{E}[X_i f(S_n)].$$

For a single summand X_1, apply a first-order Taylor expansion:

$$f(S_n) = f\Big(S_n - \frac{X_1}{\sqrt{n}}\Big) + \frac{X_1}{\sqrt{n}} f'\Big(S_n - \theta \frac{X_1}{\sqrt{n}}\Big), \quad \theta \in (0,1),$$

note that $f\Big(S_n - \frac{X_1}{\sqrt{n}}\Big)$ is independent of X_1 and that $\mathbb{E}[X_1] = 0$, so we have that

$$\mathbb{E}[X_1 f(S_n)] = \frac{1}{\sqrt{n}} \mathbb{E}\Big[X_1^2 f'\Big(S_n - \theta \frac{X_1}{\sqrt{n}}\Big)\Big].$$

Thus

$$\mathbb{E}[S_n f(S_n)] = \sqrt{n}\mathbb{E}[X_1 f(S_n)] = \mathbb{E}\Big[X_1^2 f'\Big(S_n - \theta \frac{X_1}{\sqrt{n}}\Big)\Big]$$

as $n \to \infty$. Let S be one of the subsequential limits, i.e. $S_{n_k} \to S$, we have that $\mathbb{E}\Big[f'(S_{n_k})] \to \mathbb{E}[f'(S)]$. Now, what we want in this exercise is to add some conditions such that $xf(x)$ is bounded (so that $\mathbb{E}\Big[S_{n_k} f(S_{n_k})] \to \mathbb{E}[Sf(S)]$), and that,

$$\mathbb{E}\Big[X_1^2 f'\Big(S_n - \theta \frac{X_1}{\sqrt{n}}\Big)\Big] - \mathbb{E}\Big[f'(S_n)\Big] \to 0. \tag{0.1}$$

For (0.1), we could take, e.g., Lipschitz conditions on $f'(x)$ and that X_1 has finite third moment.

Thus, we would have that

$$\mathbb{E}\Big[f'(S) - Sf(S)\Big] = 0.$$

To verify that with all these regularity conditions on f, Stein's lemma still holds, we note that it's enough to show that $S \sim N(0,1)$ by showing $\mathbb{E}\phi(S) = \mathbb{E}\phi(\chi)$ for enough different ϕ as test functions, where $\chi \sim N(0,1)$. And for each ϕ, we take a specific f such that $f'(x) - xf(x) = \phi(x) - \mathbb{E}\phi(\chi)$. In this case, we would have that $\mathbb{E}\Big[f'(S) - Sf(S)\Big] = \mathbb{E}\phi(S) - \mathbb{E}\phi(\chi)$. The solution of this ODE could be

$$f(x) := e^{\frac{x^2}{2}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} (\phi(y) - \mathbb{E}\phi(\chi)) dy.$$

Now it's not hard to find a suitable collection of test functions ϕ, so that all these f satisfy the conditions we add. We could conclude that

$$S \sim N(0,1)$$

and furthermore,

$$S_n \xrightarrow{d} N(0,1).$$

Remark: For more references, one may also see Section 6 in `https://terrytao.wordpress.com/2010/01/05/254a-notes-2-the-central-limit-theorem/#fa`.

□

Exercise 3.

Let X and Y be independent random variables with finite differential entropies $h(X)$ and $h(Y)$.
(1) ⇔ (2): Let $0 < \lambda < 1$ and define

$$W = \sqrt{\lambda}\, X + \sqrt{1-\lambda}\, Y.$$

The scaling property of differential entropy gives $h(aX) = h(X) + \log|a|$. Set $U = X/\sqrt{\lambda}$ and $V = Y/\sqrt{1-\lambda}$. Then $W = \sqrt{\lambda}\, U + \sqrt{1-\lambda}\, V$, and condition (1) becomes

$$e^{2h(W)} \geq \lambda\, e^{2h(U)} + (1-\lambda)\, e^{2h(V)}.$$

Taking logarithms and dividing by 2,

$$h(W) \geq \tfrac{1}{2} \log\big(\lambda e^{2h(U)} + (1-\lambda) e^{2h(V)}\big).$$

By concavity of log, we obtain

$$h(W) \geq \lambda h(U) + (1-\lambda) h(V),$$

which is precisely (2). Conversely, if (2) holds for all λ, apply it to U and V such that $e^{2h(U)} = e^{2h(V)}$; then (1) follows upon exponentiation. Hence, (1) and (2) are equivalent.

(1) ⇔ (3): Let X' and Y' be independent Gaussian random variables satisfying $h(X') = h(X)$ and $h(Y') = h(Y)$. Since a Gaussian Z with variance σ^2 satisfies $e^{2h(Z)} = 2\pi e\, \sigma^2$, the EPI (1) for X' and Y' holds with equality:

$$e^{2h(X'+Y')} = e^{2h(X')} + e^{2h(Y')}.$$

Thus, (1) is equivalent to

$$e^{2h(X+Y)} \geq e^{2h(X'+Y')},$$

which is the same as $h(X+Y) \geq h(X'+Y')$, i.e. statement (3).

□

Exercise 4.

(a) A simple induction on n yields

$$X^\epsilon(n\epsilon) = (1-\epsilon)^n X_0 + \sqrt{2\epsilon}\sum_{k=1}^{n}(1-\epsilon)^{n-k} Y_{k\epsilon}.$$

Since the $Y_{k\epsilon}$ are i.i.d. standard Gaussian, the second term on the right is Gaussian with mean zero and variance

$$2\epsilon\sum_{k=1}^{n}(1-\epsilon)^{2(n-k)} = 2\epsilon\frac{1-(1-\epsilon)^{2n}}{1-(1-\epsilon)^2} = 1-(1-\epsilon)^{2n}.$$

Setting $n = \lfloor t/\epsilon \rfloor$ and letting $\epsilon \downarrow 0$, we have $(1-\epsilon)^n \to e^{-t}$ and $1-(1-\epsilon)^{2n} \to 1-e^{-2t}$. Hence

$$X^\epsilon(t) \;\Rightarrow\; e^{-t}X_0 + \sqrt{1-e^{-2t}}\, Z,$$

where $Z \sim \mathcal{N}(0,1)$ is independent of X_0.

Finally, interpreting the discrete noise term as a Riemann sum,

$$\sqrt{2\epsilon}\sum_{k=1}^{n}(1-\epsilon)^{n-k}Y_{k\epsilon} \;\longrightarrow\; \sqrt{2}\int_0^t e^{-(t-s)}\, dB_s,$$

since $(1-\epsilon)^{n-k} \sim (1-\epsilon)^{(t-s)\epsilon} \to e^{-(t-s)}$. Thus

$$X_t = e^{-t}X_0 + \sqrt{2}\int_0^t e^{-(t-s)}\, dB_s.$$

(b) From Exercise 2 on Sheet 5, the density $p(t,x)$ satisfies

$$\frac{\partial}{\partial t}p(t,x) = \frac{\partial}{\partial x}\big(x\, p(t,x)\big) + \frac{\partial^2}{\partial x^2}p(t,x).$$

For any $f \in C_b^2(\mathbb{R})$ (f, f', f'' bounded, continuous), we have by differentiation under the integral and integration by parts,

$$\frac{d}{dt}\mathbb{E}[f(X_t)] = \int_{\mathbb{R}} f(x)\,\partial_t p(t,x)\,dx = \int_{\mathbb{R}} f(x)\big(\partial_x(xp) + \partial_{xx}p\big)\,dx = \mathbb{E}[f''(X_t)] - \mathbb{E}[X_t f'(X_t)].$$

□